

Characterizing graphs of maximum principal ratio

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Abstract

The principal ratio of a connected graph, denoted $\gamma(G)$, is the ratio of the maximum and minimum entries of its first eigenvector. Cioabă and Gregory conjectured that the graph on n vertices maximizing $\gamma(G)$ is a kite graph: a complete graph with a pendant path. In this paper we prove their conjecture.

1 Introduction

Several measures of graph irregularity have been proposed to evaluate how far a graph is from being regular. In this paper we determine the extremal graphs with respect to one such irregularity measure, answering a conjecture of Cioabă and Gregory [5].

All graphs in this paper will be simple and undirected, and all eigenvalues are of the adjacency matrix of the graph. For a connected graph G , the eigenvector corresponding to its largest eigenvalue, the *principal eigenvector*, can be taken to have all positive entries. If \mathbf{x} is this eigenvector, let x_{\min} and x_{\max} be the smallest and largest eigenvector entries respectively. Then define the *principal ratio*, $\gamma(G)$ to be

$$\gamma(G) = \frac{x_{\max}}{x_{\min}}.$$

Note that $\gamma(G) \geq 1$ with equality exactly when G is regular, and it therefore can be considered as a measure of graph irregularity.

Let $P_r \cdot K_s$ be the graph attained by identifying an end vertex of a path on r vertices to any vertex of a complete graph on s vertices. This has been called a *kite graph* or a *lollipop graph*. Cioabă and Gregory [5] conjectured that the connected graph on n vertices maximizing γ is a kite graph. Our main theorem proves this conjecture for n large enough.

Theorem 1. *For sufficiently large n , the connected graph G on n vertices with largest principal ratio is a kite graph.*

We note that Brightwell and Winkler [4] showed that a kite graph maximizes the expected hitting time of a random walk. Other irregularity measures for graphs have been well-studied. Bell [3] studied the irregularity measure $\epsilon(G) := \lambda_1(G) - \bar{d}(G)$, the difference between the spectral radius and the average degree of G . He determined the extremal graph over all (not necessarily connected) graphs on n vertices and e edges. It is not known what the extremal connected graph is, and Aouchiche et al [2] conjectured that this extremal graph is a ‘pineapple’:

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a complete graph with pendant vertices added to a single vertex. Bell also studied the *variance* of a graph,

$$\text{var}(G) = \frac{1}{n} \sum_{v \in V(G)} |d_v - \bar{d}|^2.$$

Albertson [1] defined a measure of irregularity by

$$\sum_{uv \in E(G)} |d(u) - d(v)|$$

and the extremal graphs were characterized by Hansen and Mélot [6].

Nikiforov [9] proved several inequalities comparing $\text{var}(G)$, $\epsilon(G)$ and $s(G) := \sum_v |d(u) - \bar{d}|$. Bell showed that $\epsilon(G)$ and $\text{var}(G)$ are incomparable in general [3]. Finally, bounds on $\gamma(G)$ have been given in [5, 10, 8, 7, 11].

2 Preliminaries

Throughout this paper G will be a connected simple graph on n vertices. The eigenvectors and eigenvalues of G are those of the adjacency matrix A of G . The vector v will be the eigenvector corresponding to the largest eigenvalue λ_1 , and we take v to be scaled so that its largest entry is 1. Let x_1 and x_k be the vertices with smallest and largest eigenvector entries respectively, and if several such vertices exist then we pick any of them arbitrarily. Let x_1, x_2, \dots, x_k be a shortest path between x_1 and x_k . Let $\gamma(G)$ be the principal ratio of G . We will abuse notation so that for any vertex x , the symbol x will refer also to $v(x)$, the value of the eigenvector entry of x . For example, with this notation the eigenvector equation becomes

$$\lambda v = \sum_{w \sim v} w.$$

We will make use of the Rayleigh quotient characterization of the largest eigenvalue of a graph,

$$\lambda_1(G) = \max_{0 \neq v} \frac{v^T A(G) v}{v^T v} \quad (1)$$

Recall that the vertices v_1, v_2, \dots, v_m are a *pendant path* if the induced graph on these vertices is a path and furthermore if, in G , v_1 has degree 1 and the vertices v_2, \dots, v_{m-1} have degree 2 (note there is no requirement on the degree of v_m).

Lemma 2. *If $\lambda_1 \geq 2$ and $\sigma = (\lambda_1 + \sqrt{\lambda_1^2 - 4})/2$, then for $1 \leq j \leq k$,*

$$\gamma(G) \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1}.$$

Moreover we have equality if the vertices x_1, x_2, \dots, x_j are a pendant path.

Proof. We have the following system of inequalities

$$\begin{array}{rcl} \lambda_1 x_1 & \geq & x_2 \\ \lambda_1 x_2 & \geq & x_1 + x_3 \\ \lambda_1 x_3 & \geq & x_2 + x_4 \\ & \vdots & \\ \lambda_1 x_{j-1} & \geq & x_j + x_{j-2} \end{array}$$

The first inequality implies that

$$x_1 \geq \frac{1}{\lambda_1} x_2$$

Plugging this into the second equation and rearranging gives

$$x_2 \geq \frac{\lambda_1}{\lambda_1^2 - 1} x_3$$

Now assume that

$$x_i \geq \frac{u_{i-1}}{u_i} x_{i+1}.$$

with u_j positive for all $j < i$. Then

$$\lambda_1 x_{i+1} \geq x_i + x_{i+2}$$

implies that

$$x_{i+1} \geq \frac{u_i}{\lambda_1 u_i - u_{i-1}} x_{i+2}.$$

where $\lambda_1 u_i - u_{i-1}$ must be positive because x_j is positive for all j . Therefore the coefficients u_i satisfy the recurrence

$$u_{i+1} = \lambda_1 u_i - u_{i-1}$$

Solving this and using the initial conditions $u_0 = 1$, $u_1 = \lambda$ we get

$$u_i = \frac{\sigma^{i+1} - \sigma^{-i-1}}{\sigma - \sigma^{-1}}$$

In particular, u_i is always positive, a fact implicitly used above. Finally this gives,

$$x_1 \geq \frac{u_0}{u_1} x_2 \geq \frac{u_0}{u_1} \cdot \frac{u_1}{u_2} x_3 \geq \cdots \geq \frac{x_j}{u_{j-1}}$$

Hence

$$\gamma(G) = \frac{x_k}{x_1} = \frac{1}{x_1} \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1}$$

If these vertices are a pendant path, then we have equality throughout. \square

We will also use the following lemma which comes from the paper of Cioabă and Gregory [5].

Lemma 3. *For $r \geq 2$ and $s \geq 3$,*

$$s - 1 + \frac{1}{s(s-1)} < \lambda_1(P_r \cdot K_s) < s - 1 + \frac{1}{(s-1)^2}.$$

In the remainder of the paper we prove Theorem 1. We now give a sketch of the proof that is contained in Section 3.

1. We show that the vertices x_1, x_2, \dots, x_{k-2} are a pendant path and that x_k is connected to all of the vertices in G that are not on this path (lemma 5).
2. Next we prove that the length of the path is approximately $n - n/\log(n)$ (lemma 6).
3. We show that x_{k-2} has degree exactly 2 (lemma 9), which extends our pendant path to x_1, x_2, \dots, x_{k-1} . To do this, we find conditions under which adding or deleting edges increases the principal ratio (lemma 7).
4. Next we show that x_{k-1} also has degree exactly 2 (lemma 11). At this point we can deduce that our extremal graph is either a kite graph or a graph obtained from a kite graph by removing some edges from the clique. We show that adding in any missing edges will increase the principal ratio, and hence the extremal graph is exactly a kite graph.

3 Proof of Theorem 1

Let G be the graph with maximal principal ratio among all connected graphs on n vertices, and let k be the number of vertices in a shortest path between the vertices with smallest and largest eigenvalue entries. As above, let x_1, \dots, x_k be the vertices of the shortest path, where $\gamma(G) = x_k/x_1$. Let C be the set of vertices not on this shortest path, so $|C| = n - k$. Note that there is no graph with $n - k = 1$, as the endpoints of a path have the same principal eigenvector entry. Also $\lambda_1(G) \geq 2$, otherwise $P_{n-2} \cdot K_3$ would have larger principal ratio. Finally note that k is strictly larger than 1, otherwise $x_k = x_1$ and G would be regular.

Lemma 4. $\lambda_1(G) > n - k$.

Proof. Let H be the graph $P_k \cdot K_{n-k+1}$. It is straightforward to see that in H , the smallest entry of the principal eigenvector is the vertex of degree 1 and the largest is the vertex of degree $n - k + 1$. Also note that in H , the vertices on the path P_k form a pendant path. By maximality we know that $\gamma(G) \geq \gamma(H)$. Combining this with lemma 2, we get

$$\frac{\sigma^k - \sigma^{-k}}{\sigma - \sigma^{-1}} \geq \gamma(G) \geq \gamma(H) = \frac{\sigma_H^k - \sigma_H^{-k}}{\sigma_H - \sigma_H^{-1}}$$

where $\sigma_H = (\lambda_1(H) + \sqrt{\lambda_1(H)^2 - 4})/2$.

Now the function

$$f(x) = \frac{x^k - x^{-k}}{x - x^{-1}}$$

is increasing when $x \geq 1$. Hence we have $\sigma \geq \sigma_H$, and so $\lambda_1(G) \geq \lambda_1(H) > n - k$. \square

Lemma 5. x_1, x_2, \dots, x_{k-2} are a pendant path in G , and x_k is connected to every vertex in G that is not on this path.

Proof. By our choice of scaling, $x_k = 1$. From lemma 4

$$n - k < \lambda_1(G) = \sum_{y \sim x_k} y \leq |N(x_k)|.$$

Now $|N(x_k)|$ is an integer, so we have $|N(x_k)| \geq n - k + 1$. Moreover because x_1, x_2, \dots, x_k is an induced path, we must have that $|N(x_k)| = n - k + 1$ exactly, and hence the $N(x_k) = C \cup \{x_{k-1}\}$. It follows that x_1, x_2, \dots, x_{k-3} have no neighbors off the path, as otherwise there would be a shorter path between x_1 and x_k . \square

Lemma 6. For the extremal graph G , we have $n - k = (1 + o(1)) \frac{n}{\log n}$.

Proof. Let H be the graph $P_j \cdot K_{n-j+1}$ where $j = \left\lfloor n - \frac{n}{\log n} \right\rfloor$, and let G be the connected graph on n vertices with maximum principal ratio. Let x_1, \dots, x_k be a shortest path from x_1 to x_k where $\gamma(G) = \frac{x_k}{x_1}$. By lemma 5, we have

$$\lambda_1(G) \leq \Delta(G) \leq n - k + 1.$$

By the eigenvector equation, this gives that

$$\gamma(G) \leq (n - k + 1)^k \tag{2}$$

Now, lemma 2 gives that

$$\gamma(H) = \frac{\sigma_H^j - \sigma_H^{-j}}{\sigma_H - \sigma_H^{-1}},$$

where

$$\sigma(H) = \frac{\lambda_1(H) + \sqrt{\lambda_1(H)^2 - 4}}{2}.$$

Now, $s - 1 + \frac{1}{s(s-1)} < \lambda_1(P_r \cdot K_s) < s - 1 + \frac{1}{(s-1)^2}$, so we may choose n large enough that $\frac{n}{\log n} + 1 > \sigma_H - \sigma_H^{-1} > \frac{n}{\log n}$. By maximality of $\gamma(G)$, we have

$$(n - k + 1)^k \geq \gamma(G) \geq \gamma(H) \geq \left(\frac{n}{\log n} \right)^{n - \frac{n}{\log n} - 2}.$$

Thus, $n - k = (1 + o(1)) \frac{n}{\log n}$. □

For the remainder of this paper we will explore the structure of G by showing that if certain edges are missing, adding them would increase the principal ratio, and so by maximality these edges must already be present in G . We have established that the vertices x_1, x_2, \dots, x_{k-2} are a pendant path, and so we have

$$\gamma(G) = \frac{\sigma^{k-2} - \sigma^{-k+2}}{\sigma - \sigma^{-1}} \frac{1}{x_{k-2}} \quad (3)$$

We will not add any edges that affect this path, and so the above equality will remain true. The change in γ is then completely determined by the change in λ_1 and the change in x_{k-2} . The next lemma gives conditions on these two parameters under which γ will increase or decrease.

Lemma 7. *Let x_1, x_2, \dots, x_{m-1} form a pendant path in G , where $n - m = (1 + o(1))n/\log(n)$. Let G_+ be a graph obtained from G by adding some edges from x_{m-1} to $V(G) \setminus \{x_1, \dots, x_{m-1}\}$, where the addition of these edges does not affect which vertex has largest principal eigenvector entry. Let λ_1^+ be the largest eigenvalue of G_+ with leading eigenvector entry for vertex x denoted x^+ , also normalized to have maximum entry one. Define δ_1 and δ_2 such that $\lambda_1^+ = (1 + \delta_1)\lambda_1$ and $x_{m-1}^+ = (1 + \delta_2)x_{m-1}$. Then*

- $\gamma(G_+) > \gamma(G)$ whenever $\delta_1 > 4\delta_2/n$
- $\gamma(G_+) < \gamma(G)$ whenever $\delta_1 \exp(2\delta_1 \lambda_1 \log n) < \delta_2/3n$.

Proof. We have

$$\sigma = \lambda_1 - \lambda_1^{-1} - \lambda_1^{-3} - 2\lambda_1^{-5} - \dots - \frac{2}{2n-3} \binom{2n-2}{n} \lambda_1^{-(2n-1)} - \dots$$

So

$$\lambda_1^+ - \lambda_1 < \sigma_+ - \sigma < \lambda_1^+ - \lambda_1 - 2((\lambda_1^+)^{-1} - \lambda_1^{-1})$$

when λ_1 is sufficiently large, which is guaranteed by lemma 6. Plugging in $\lambda_1^+ = (1 + \delta_1)\lambda_1$, we get

$$\delta_1 \lambda_1 < \sigma_+ - \sigma < \delta_1 \lambda_1 + 2\lambda_1^{-1}(1 - (1 + \delta_1)^{-1}) < \delta_1 \lambda_1 + \delta_1$$

In particular

$$(1 + \delta_1/2)\sigma < \sigma_+ < (1 + 2\delta_1)\sigma$$

To prove part (i), we wish to find a lower bound in the change in the first factor of equation 3. Let

$$f(x) = \frac{x^{m-1} - x^{-m+1}}{x - x^{-1}}.$$

Then $2mx^{m-3} > f'(x) > (m-2)x^{m-3} - mx^{m-5}$, and using that $n-m \sim n/\log(n)$ and $\sigma \sim \lambda_1$ which goes to infinity with n , we get $f'(x) \gtrsim (m-2)x^{m-3}$. By linearization and because $f(\sigma) \sim \sigma^{m-2}$, it follows that

$$\frac{\sigma_+^{m-1} - \sigma_+^{-m+1}}{\sigma_+ - \sigma_+^{-1}} \geq \left(1 + \frac{\delta_1(m-3)}{2}\right) \frac{\sigma^{m-1} - \sigma^{-m+1}}{\sigma - \sigma^{-1}}$$

Hence, if

$$\frac{\delta_1(m-3)}{2} > \delta_2$$

then $\gamma(G_+) > \gamma(G)$. In particular it is sufficient that $\delta_1 > 4\delta_2/n$.

To prove part (ii), recall from above that $f'(x) < 2mx^{m-3}$. Then, when $x = (1 + o(1))(n/\log(n))$

$$\begin{aligned} f'(x + \varepsilon) &< 2m(x + \varepsilon)^{m-3} \\ &= 2mx^{m-3} \left(1 + \frac{\varepsilon}{x}\right)^{m-3} \\ &\leq 2mx^{m-3} \exp\left(\frac{m\varepsilon}{x}\right) \\ &\leq 2nx^{m-3} \exp(2\log(n)\varepsilon) \end{aligned}$$

So for $0 < \varepsilon < \delta_1 \lambda_1$, we have

$$f'(x + \varepsilon) < 2nx^{m-3} \exp(2\log(n)\delta_1 \lambda_1)$$

Hence

$$(1 + 3n \exp(2\delta_1 \lambda_1 \log n) \delta_1) \frac{\sigma^{m-1} - \sigma^{-m+1}}{\sigma - \sigma^{-1}} > \frac{\sigma_+^{m-1} - \sigma_+^{-m+1}}{\sigma_+ - \sigma_+^{-1}}$$

□

Lemma 8. *For every subset of U of $N(x_k)$, we have*

$$|U| - 1 < \sum_{y \in U} y \leq |U|.$$

An immediate consequence is that there is at most one vertex in the neighborhood of x_k with eigenvector entry smaller than $1/2$.

Proof. The upper bound follows from $y \leq 1$, and the lower bound from the inequalities

$$\sum_{y \in N(x_k) \setminus U} y \leq |N(x_k)| - |U|$$

and

$$\sum_{y \in N(x_k)} y = \lambda_1(G) > |N(x_k)| - 1.$$

□

Lemma 9. *The vertex x_{k-2} has degree exactly 2 in G .*

Proof. Assume to the contrary. Let $U = N(x_{k-2}) \cap N(x_k)$. Then $|U| \geq 2$, so by lemma 8 we have

$$\sum_{y \in U} y > |U| - 1 \geq 1.$$

Now, by the same argument as the in the proof of lemma 2, we have that

$$\gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \left(\sum_{y \in U} y \right)^{-1}$$

Let $H = P_{k-1} \cdot K_{n-k+2}$. Then by maximality of $\gamma(G)$ we have

$$\frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} > \gamma(G) \geq \gamma(H) = \frac{\sigma_H^{k-1} - \sigma_H^{-k+1}}{\sigma_H - \sigma_H^{-1}}$$

So $\sigma > \sigma_H$, which means $\lambda_1(G) > \lambda_1(H) > n - k + 1$. This means that $\Delta(G) > n - k + 1$, but we have established that $\Delta(G) = n - k + 1$. \square

We now know that x_1, x_2, \dots, x_{k-1} is a pendant path in G , and so equation 3 becomes

$$\gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \frac{1}{x_{k-1}} \quad (4)$$

Lemma 10. *The vertex x_{k-1} has degree less than $11|C|/\sqrt{\log n}$.*

Proof. Assume to the contrary, so throughout this proof we assume that the degree of x_{k-1} is at least $11|C|/\sqrt{\log n}$. Let G_+ the graph obtained from G with an additional edge from x_{k-1} to a vertex $z \in C$ with $z \geq 1/2$. Let $\lambda_1^+ = \lambda_1(G_+)$ and let x^+ be the principal eigenvector entry of vertex x in G_+ , where this eigenvector is normalized to have $x_k^+ = 1$.

Change in λ_1 : By equation 1, we have $\lambda_1^+ - \lambda_1 \geq 2 \frac{x_{k-1}z}{\|v\|_2^2}$. A crude upper bound on $\|v\|_2^2$ is

$$\|v\|_2^2 \leq 1 + \sum_{y \sim x_k} y + \frac{2}{\lambda_1} + \frac{4}{\lambda_1^2} + \dots < 2\lambda_1$$

We also have that $z \geq 1/2$ so

$$\lambda_1^+ \geq \left(1 + \frac{x_{k-1}}{2\lambda_1^2} \right) \lambda_1.$$

Change in x_{k-1} : Let $U = N(x_{k-1} \cap C)$. By the eigenvector equation we have

$$\begin{aligned} x_{k-1} &= \frac{1}{\lambda_1} \left(x_{k-2} + x_k + \sum_{y \in U} y \right) \\ x_{k-1}^+ &= \frac{1}{\lambda_1^+} \left(x_{k-2}^+ + x_k^+ + z^+ + \sum_{y \in U} y^+ \right) \end{aligned}$$

Subtracting these, and using that $\lambda_1 < \lambda_1^+$ and $x_k = x_k^+ = 1$, we get

$$x_{k-1}^+ - x_{k-1} \leq \frac{1}{\lambda_1} \left(x_{k-2}^+ - x_{k-2} + z^+ + \sum_{y \in U} y^+ - y \right).$$

By lemma 8, we have $\sum_{y \in U} y^+ - y \leq 1$. We also have $x_{k-2}^+ - x_{k-2} < 1$ and $z^+ \leq 1$. Hence $x_{k-1}^+ - x_{k-1} \leq 3/\lambda_1$, or

$$x_{k-1}^+ \geq \left(1 + \frac{3}{\lambda_1 x_{k-1}}\right) x_{k-1}$$

We can only apply lemma 7 if x_k^+ is the largest eigenvector entry in G_+ . So we must consider two cases.

Case 1: If in G^+ the largest eigenvector entry is still attained by vertex x_k , then we can apply lemma 7, and see that $\gamma(G^+) > \gamma(G)$ if

$$\frac{x_{k-1}}{2\lambda_1^2} \geq \frac{12}{\lambda_1 x_{k-1} n}$$

or equivalently

$$x_{k-1}^2 \geq \frac{24\lambda_1}{n}.$$

We have that $\lambda_1 = (1 + o(1))(n - n/\log(n))$, so it suffices for

$$x_{k-1} \geq \frac{5}{\sqrt{\log n}}. \quad (5)$$

We know that

$$x_{k-1} > \frac{|U| - 1}{2\lambda_1}.$$

By assumption

$$|U| + 2 = N(x_{k-1}) \geq 11|C|/\sqrt{\log n}$$

Equation 5 follows from this, so $\gamma(G^+) > \gamma(G)$.

Case 2: Say the largest eigenvector entry of G^+ is no longer attained by vertex x_k . It is easy to see that the largest eigenvector entry is not attained by a vertex with degree less than or equal to 2, and comparing the neighborhood of any vertex in C with the neighborhood of x_k we can see that $x_k \geq y$ for all $y \in C$. So the largest eigenvector entry must be attained by x_{k-1} . Then equation 4 no longer holds, instead we have

$$\gamma(G_+) = \frac{\sigma_+^{k-1} - \sigma_+^{-k+1}}{\sigma_+ - \sigma_+^{-1}}. \quad (6)$$

Recall that in lemma 7 we determined the change from $\gamma(G_+)$ to $\gamma(G)$ by considering $\lambda_1^+ - \lambda_1$ and $x_{k-1}^+ - x_{k-1}$. In this case, by (6), we must consider $\lambda_1^+ - \lambda_1$ and $1 - x_{k-1}$. Now if $x_{k-1}^+ > x_k^+$, then vertex x_{k-1} in G is connected to all of C except perhaps a single vertex. Hence in G , the vertex x_{k-1} is connected to all of C except at most two vertices. This gives the bound

$$1 - x_{k-1} \leq 3/\lambda_1$$

and so as in the previous case, $\gamma(G_+) > \gamma(G)$.

So in all cases, x_{k-1} is connected to all vertices in C that have eigenvector entry larger than $1/2$. If all vertices in C have eigenvector entry larger than $1/2$, then x_{k-1} is connected to all of C , and this implies that $x_{k-1} > x_k$, which is a contradiction. At most one vertex in C is smaller than $1/2$, and so there is a single vertex $z \in C$ with $z < 1/2$. We will quickly check that adding the edge $\{x_{k-1}, z\}$ increases the principal ratio. As before let G_+ be the graph obtained by adding this edge. The largest eigenvector entry in G_+ is attained by x_{k-1} , as its neighborhood strictly contains the neighborhood of x_k . As above, adding the edge $\{z, x_k\}$ increases the spectral radius at least

$$\lambda_1^+ > \left(1 + \frac{z}{2\lambda_1^2}\right) \lambda_1$$

and we have $1 - x_{k-1} < 1 - z/\lambda_1$. Applying lemma 7 we see that $\gamma(G_+) > \gamma(G)$, which is a contradiction. Finally we conclude that the degree of x_{k-1} must be smaller than $11|C|/\sqrt{\log n}$. \square

We note that this lemma gives that $x_{k-1} < 1/2$ which implies that any vertex in C has eigenvector entry larger than $1/2$.

Lemma 11. *The vertex x_{k-1} has degree exactly 2 in G . It follows that $x_{k-1} < 2/\lambda_1$.*

Proof. Let $U = N(x_{k-1}) \cap C$, $c = |U|$. If $c = 0$ then we are done. Otherwise let G_- be the graph obtained from G by deleting these C edges. We will show that $\gamma(G_-) > \gamma(G)$.

(1) Change in λ_1 : We have by equation 1,

$$\lambda_1 - \lambda_1^- \leq 2c \frac{x_{k-1}}{\|v\|_2^2}$$

By Cauchy-Schwarz,

$$\|v\|_2^2 > \sum_{x \in N(x_k)} x^2 \geq \frac{\left(\sum_{x \in N(x_k)} x\right)^2}{|C| + 1} \geq \frac{(n-k)^2}{n-k+1}$$

We also have

$$x_{k-1} \leq \frac{c+2}{\lambda_1}$$

Combining these we get

$$\lambda_1 - \lambda_1^- < \frac{9c^2}{\lambda_1(n-k+1)} \Rightarrow \lambda_1 < \left(1 + \frac{9c^2}{\lambda_1 \lambda_1^- (n-k+1)}\right) \lambda_1^-$$

We have $\lambda_1 \lambda_1^- > (n-k)^2$, so

$$\lambda_1 < \left(1 + \frac{10c^2}{(n-k)^3}\right) \lambda_1^-$$

(2) Change in x_{k-1} : At this point, we know that in G_- the vertices x_1, \dots, x_k form a pendant path, and so by the proof of lemma 2, we have $x_{k-1}^- = (1+o(1))/\lambda_1$. By the eigenvector equation and using that the vertices in C have eigenvector entry at least $1/2$, we have $x_{k-1} > (1+c/2)/\lambda_1$. So

$$x_{k-1} - x_{k-1}^- > \frac{1}{\lambda_1} \left(\frac{c}{2} + o(1)\right)$$

In particular,

$$x_{k-1} > \left(1 + \frac{c}{3x_{k-1}^- \lambda_1}\right) x_{k-1}^-$$

Applying lemma 7, it suffices now to show that

$$\frac{10c^2}{(n-k)^3} \exp\left(2 \frac{10c^2}{(n-k)^3} \lambda_1^- \log n\right) < \frac{c}{9x_{k-1}^- \lambda_1 n}. \quad (7)$$

Now

$$\frac{10c^2}{(n-k)^3} < 10 \frac{11^2}{\log(n)} \frac{|C|^2}{(n-k)^3} < \frac{11^3}{\log n} \frac{\log n}{n} = \frac{11^3}{n}.$$

Similarly $2\frac{10c^2}{(n-k)^3}\lambda_1^-\log n < 2 \cdot 11^3$, so the lefthand side of equation 7 is smaller than C_0/n , where C_0 is an absolute constant. For the righthand side, recall that $x_{k-1}^-\lambda_1 = 1 + o(1)$, and also that

$$c > \frac{11}{\sqrt{\log n}} \left(\frac{n}{\log n} + o(1) \right) > \frac{10n}{\log^{3/2} n}.$$

So the righthand side is larger than $1/\log^{3/2} n$. Hence for large enough n , the righthand side is larger than the lefthand side. \square

We are now ready to prove the main theorem.

Theorem 1. *For sufficiently large n , the connected graph G on n vertices with largest principal ratio is a kite graph.*

Proof. It remains to show that C induces a clique. Assume it does not, and let H be the graph $P_k \cdot K_{n-k+1}$. We will show that $\gamma(H) > \gamma(G)$, and this contradiction tells us that C is a clique. As before, lemma 2 gives that

$$\gamma(H) = \frac{\sigma_H^k - \sigma_H^{-k}}{\sigma_H - \sigma_H^{-1}},$$

where

$$\sigma(H) = \frac{\lambda_1(H) - \sqrt{\lambda_1(H)^2 - 4}}{2}.$$

Since x_1, \dots, x_k form a pendant path we also know that

$$\gamma(G) = \frac{\sigma^k - \sigma^{-k}}{\sigma - \sigma^{-1}}.$$

Now, $\lambda_1(H) > \lambda_1(G)$ because $E(G) \subsetneq E(H)$. Since the functions $g(x) = x + \sqrt{x^2 - 4}$ and $f(x) = (x^k - x^{-k})/(x - x^{-1})$ are increasing when $x \geq 1$, we have $\gamma(H) > \gamma(G)$. \square

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